ON THE DEGREE OF CONVERGENCE OF STURM-LIOUVILLE SERIES*

BY

DUNHAM JACKSON

After the degree of convergence of Fourier's series has been studied to a certain extent, as has been done by various authors, including the present writer, it is natural to inquire whether the results obtained are capable of extension to other series of characteristic solutions of homogeneous linear differential equations. A theorem in this connection has already been published by Tamarkine.† It is the purpose of the present paper to develop a greater variety of facts relating to a much less general differential equation than that which Tamarkine treats. In discussing expansion problems connected with an equation of the second order, Kneser,‡ following Liouville,§ establishes relations from which it is at once apparent that the theorem of Lebesgue,|| which states that the remainder after n terms in the Fourier's series for a function satisfying a Lipschitz condition does not exceed a constant multiple of $(\log n)/n$, applies equally well to the development of such a function in Sturm-Liouville series of the form under consideration. It is to be shown how these relations can be refined so as to make it possible to carry over other theorems of the same nature from the Fourier's series to the more general one. The theorems susceptible of this generalization include that of Picard* based on the hypothesis that the function developed has a kth derivative of limited variation, and one proved by the author† for a function having

* Presented to the Society, December 30, 1913.

† Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), pp. 345–382; see especially p. 368. In connection with this paper, though not with reference to the particular point mentioned here, see also, in the same journal, Birkhoff, vol. 36 (1913), pp. 115–126, and Tamarkine, vol. 37 (1914), pp. 376–378.

‡ Mathematische Annalen, vol. 58 (1904), pp. 81–147; see p. 127.

§ Journal de mathématiques pures et appliquées, vol. 2 (1837), pp. 418–436.

|| Bulletin de la société mathématique de France, vol. 38 (1910), pp. 184–210; p. 201.

* *Traité d'Analyse*, vol. 1, chapter IX, § 12. Picard is not directly concerned with the degree of convergence of the series, but nothing more is needed in this case than the formulas which he gives for the order of magnitude of the coefficients. The hypothesis concerning the function also is not exactly that stated here, but is essentially the same for the purposes of the proof. When the Lipschitz condition is used a knowledge of the order of magnitude of the coefficients is not enough; see Lebesgue, loc. cit., pp. 192–195.

† These Transactions, vol. 13 (1912), pp. 491–515, Theorem V. This paper will be referred to as A.

a $(k-1)$th derivative that satisfies a Lipschitz condition. The case of a function having a finite second derivative, which was treated by Liouville,* is included in that of functions having a first derivative of limited variation. The method employed is intimately related with that by which Horn† obtains extended asymptotic expressions for solutions of the differential equation.

For the proofs relating to the higher degrees of convergence, additional hypotheses, beyond that of mere continuity, are made concerning the functions which appear as coefficients in the differential equation, and it is assumed that the function developed vanishes with a sufficient number of its derivatives at the ends of the interval. The latter requirement may seem unduly restrictive, but an examination of the facts in the simplest case, that of the cosine-series,‡ shows that a limitation of this nature is required for the truth of the conclusions, and is not due merely to inadequacy of the method of treatment.

In the last part of the paper attention is given to the problem of representing a function that has a $(k-1)$th derivative satisfying a Lipschitz condition by means of a linear combination of a finite number of characteristic functions, with a higher degree of approximation than that afforded by the sum of the corresponding terms in the Sturm-Liouville series itself. Such a representation is obtained by summing the series by a method which the author had previously applied to Fourier's series.

1. Preliminary Statements

The differential equation with which we shall deal is the following:

$$\frac{d^2U}{dx^2}+[\rho^2-\lambda(x)]\,U=0. \tag{1}$$

Here $\lambda(x)$ is a function which is assumed at the outset to be continuous, and will be subjected to further restrictions as occasion demands, but only when such restrictions are explicitly mentioned. The parameter ρ^2 is not restricted to positive nor even to real values. The familiar transformation§ which reduces a more general equation to this form gives so directly the interpretation of our results with reference to the general equation that it is unnecessary to dwell upon the latter. The boundary conditions are

$$\begin{aligned} U'(0)-h'\,U(0)&=0,\\ U'(\pi)+H'\,U(\pi)&=0, \end{aligned} \tag{2}$$

* Journal de Mathématiques pures et appliquées, vol. 2 (1837) pp. 16–35; see especially the sentence beginning at the foot of p. 18. See also Kneser, loc. cit., pp. 121–123.

† Mathematische Annalen, vol. 52 (1899), pp. 271–292.

‡ It is the cosine-series, rather than the complete Fourier's series, which is the true prototype of the expansions considered in this article.

§ See, e. g., Kneser, loc. cit., pp. 116–117.

the interval over which the variable x is to range being taken for convenience as that from 0 to π. The numbers h' and H' are real constants, not restricted as to sign; the notation is the classical one. The limiting cases obtained by letting one or both of these constants become infinite, that is, by making the solution U vanish at one or both ends of the interval, while requiring a discussion differing in details from the one that is to be given, offer nothing essentially new that is not provided for in the paper of Kneser already cited, and will not be treated separately here.

The following facts concerning the characteristic numbers and solutions of the system consisting of the equation (1) and the boundary conditions (2) will be assumed as well known.

There are infinitely many real* values of ρ^2 for which the system has a real solution not identically zero. Only a finite number of these values can be negative, and they have no cluster-point in the finite plane. If ρ_n represents the positive square root of the nth of them in algebraic order of magnitude, when n is large enough so that the corresponding value of ρ^2 is positive, then

(3) $$\rho_n = n + \epsilon_n,$$

where†

(4) $$\epsilon_n = O\left(\frac{1}{n}\right).$$

Two linearly independent solutions can not correspond to the same value of ρ^2. If the arbitrary constant factor in the nth characteristic solution $U_n(x)$ is suitably determined, the solution satisfies the integral equation

(5) $$U_n(x) = \cos \rho_n x + \frac{h' \sin \rho_n x}{\rho_n} + \frac{1}{\rho_n}\int_0^x \lambda(t)\, U_n(t) \sin \rho_n (x - t)\, dt,$$

when the corresponding value of ρ^2 is positive. The maximum value of $U_n(x)$ in the interval $(0, \pi)$ remains finite as n becomes infinite,‡ so that the third as well as the second term on the right-hand side is very small when n is very large.

We shall be interested in the degree of convergence of the series

(6) $$\sum_{n=0}^{\infty} \alpha_n U_n(x),$$

where

(7) $$\alpha_n = \frac{\int_0^\pi \phi(x)\, U_n(x)\, dx}{\int_0^\pi [U_n(x)]^2\, dx},$$

* The fact that there are no complex characteristic values will not be used explicitly.

† This notation means that the absolute value of ϵ_n does not exceed a constant multiple of $1/n$.

‡ This is readily deduced from (5) itself.

and $\phi(x)$ is a continuous function on which further restrictions will be imposed later. It will be assumed as known* that if this series converges uniformly, its value must necessarily be $\phi(x)$.

With this review of facts that are presupposed,† we go on to the detailed discussion of the problem in hand.

2. Degree of Convergence of the Series

The study of the degree of convergence of the series (6) involves an examination of the integral

$$\int_0^\pi \phi(x)\, U_n(x)\, dx, \tag{8}$$

which appears in the numerator of the expression (7) for the general coefficient. In evaluating this integral approximately, use will be made of the formula (5) for $U_n(x)$; but this identity will be written at greater length by substituting the whole right-hand side for the function U_n under the sign of integration, and repeating this process a number of times. It will be well to anticipate the discussion of the whole integral (8) by three lemmas relating to the individual terms that will be obtained.

Lemma I. *If $\phi(x)$ has a continuous‡ kth derivative of limited variation in the interval $0 \leqq x \leqq \pi$, and if*

$$\begin{aligned}\phi(0) = \phi'(0) = \cdots = \phi^{(k-1)}(0)&\\ = \phi(\pi) = \phi'(\pi) = \cdots = \phi^{(k-1)}(\pi) = 0&,\end{aligned} \tag{9}$$

then§

$$\int_0^\pi \phi(x) \cos \rho_n x\, dx = O\left(\frac{1}{n^{k+1}}\right), \tag{10}$$

and

$$\int_0^\pi \phi(x) \sin \rho_n x\, dx = O\left(\frac{1}{n^{k+1}}\right). \tag{11}$$

Consider first the integral containing $\cos \rho_n x$ in the integrand. Since ϕ vanishes at both ends of the interval, integration by parts gives at first

$$\int_0^\pi \phi(x) \cos \rho_n x\, dx = -\frac{1}{\rho_n}\int_0^\pi \phi'(x) \sin \rho_n x\, dx,$$

and after a sufficient number of repetitions,

$$\int_0^\pi \phi(x) \cos \rho_n x\, dx = \frac{1}{\rho_n^k}\int_0^\pi \phi^{(k)}(x) \cos\left(\rho_n x + \frac{k\pi}{2}\right) dx. \tag{12}$$

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* See, e. g., Kneser, loc. cit., pp. 109–116, 123–124.

† For a concise exposition of the properties of functions of limited variation that will be used, see, e. g., E. B. Wilson, *Advanced Calculus*, pp. 309, 310.

‡ The assumption of continuity is not necessary, but is made for the sake of convenience.

§ Cf. Picard, loc. cit., where the corresponding proof is given for the case of the Fourier's series.

As the hypothesis is that $\phi^{(k)}(x)$ is of limited variation, let

$$\phi^{(k)}(x) = \phi_1(x) - \phi_2(x), \tag{13}$$

where ϕ_1 and ϕ_2 are positive or zero, continuous, and monotone increasing. By the second law of the mean,

$$\int_0^\pi \phi^{(k)}(x)\cos\left(\rho_n x + \frac{k\pi}{2}\right)dx = \phi_1(\pi)\int_\xi^\pi \cos\left(\rho_n x + \frac{k\pi}{2}\right)dx - \phi_2(\pi)\int_\eta^\pi \cos\left(\rho_n x + \frac{k\pi}{2}\right)dx,$$

where ξ and η are numbers in the interval $(0, \pi)$. The absolute value of the expression just written down does not exceed

$$\frac{2}{\rho_n}[\phi_1(\pi) + \phi_2(\pi)].$$

Since, by (3) and (4), ρ_n is of the same order of magnitude as n, the relation (10) follows at once. The proof of (11) is precisely similar.

If $\phi^{(k)}(0) = 0$, it may be assumed that $\phi_1(0) = \phi_2(0) = 0$, and that the total variation v of $\phi^{(k)}(x)$ is equal to $\phi_1(\pi) + \phi_2(\pi)$. In this case, either of the integrals in the lemma is at most equal to $2v/\rho_n^{k+1}$. It will be convenient to use the letter c a number of times as a general notation for a positive constant, sometimes one and sometimes another, which is independent of n, x, and the function ϕ, though it may depend, for example, on the coefficient λ in the differential equation and on the coefficients h' and H' in the boundary conditions. With this convention, we may say that if $\phi^{(k)}(0) = 0$ each of the integrals remains inferior in absolute value to* cv/n^{k+1}.

Another interesting remark of a special nature is that when $k = 1$ the hypothesis (9), reducing here to

$$\phi(0) = \phi(\pi) = 0,$$

may be abandoned as far as the relation (10) is concerned. For

$$\int_0^\pi \phi(x)\cos\rho_n x\,dx = \frac{1}{\rho_n}[\phi(x)\sin\rho_n x]_0^\pi - \frac{1}{\rho_n}\int_0^\pi \phi'(x)\sin\rho_n x\,dx,$$

and $\sin\rho_n x$ vanishes when $x = 0$, while

$$\sin\rho_n\pi = \pm\sin\epsilon_n\pi = O\left(\frac{1}{n}\right),$$

by (4).

* This is more precise than the original statement of the lemma, because there it was not specified how the constant multiplier implied in the O-notation depends on the function ϕ; here we see that it can be taken proportional to the total variation of $\phi^{(k)}$.

LEMMA II. *If $\phi(x)$ satisfies the conditions of Lemma I, and if $\lambda(x)$ has a continuous $(k-1)$th derivative with limited variation** *in* $0 \leqq x \leqq \pi$, *then*

$$\int_0^\pi \phi(x) \int_0^x \lambda(t_1) \cos \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1\, dx = O\left(\frac{1}{n^{k+1}}\right), \tag{14}$$

$$\int_0^\pi \phi(x) \int_0^x \lambda(t_1) \sin \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1\, dx = O\left(\frac{1}{n^{k+1}}\right). \tag{15}$$

By inversion of the order of integration the first of these two integrals may be given the form

$$\int_0^\pi \lambda(t_1) \cos \rho_n t_1 \int_0^\pi \phi(x) \sin \rho_n (x - t_1)\, dx\, dt_1.$$

Let the inner integral here be transformed by integration by parts. The function $\phi(x)$ vanishes with its first $k-1$ derivatives when $x = \pi$, but of course not, in general, when $x = t_1$. At the latter point, the sine of $\rho_n(x - t_1)$ vanishes, while the cosine is equal to 1. Hence

$$\begin{aligned}\int_{t_1}^\pi \phi(x) \sin \rho_n (x - t_1)\, dx &= \frac{1}{\rho_n}\phi(t_1) + \frac{1}{\rho_n}\int_{t_1}^\pi \phi'(x) \cos \rho_n (x - t_1)\, dx \\ &= \frac{1}{\rho_n}\phi(t_1) - \frac{1}{\rho_n^2}\int_{t_1}^\pi \phi''(x) \sin \rho_n (x - t_1)\, dx,\end{aligned}$$

and so on. Finally, if we let $k = 2\gamma$ or $2\gamma - 1$, according as k is even or odd,

$$\begin{aligned}\int_{t_1}^\pi \phi(x) \sin \rho_n (x - t_1)\, dx &= \frac{1}{\rho_n}\phi(t_1) - \frac{1}{\rho_n^3}\phi''(t_1) + \cdots \\ &+ \frac{(-1)^{\gamma-1}}{\rho_n^{2\gamma-1}}\phi^{(2\gamma-2)}(t_1) + \frac{1}{\rho_n^k}\int_{t_1}^\pi \phi^{(k)}(x) \sin\left[\rho_n(x - t_1) + \frac{k\pi}{2}\right] dx.\end{aligned} \tag{16}$$

By the use of (13) and the second law of the mean, it is seen that the absolute value of the last integral does not exceed

$$\frac{2}{\rho_n}[\phi_1(\pi) + \phi_2(\pi)],$$

whatever the value of t_1 may be.

We have now to multiply the several terms of (16) by $\lambda(t_1) \cos \rho_n t_1$ and integrate with respect to t_1 from 0 to π. The last term, in consequence of what we have just seen, will yield a quantity which is in order of magnitude $O(1/\rho_n^{k+1})$, and therefore $O(1/n^{k+1})$.

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* The function $\lambda(x)$ is not subjected to any special restrictions at the ends of the interval.

Consider the first term of the integrated expression,

$$\frac{1}{\rho_n}\int_0^\pi \lambda(t_1)\phi(t_1)\cos\rho_n t_1\,dt_1. \tag{17}$$

The function $\lambda(t_1)\phi(t_1)$ has a $(k-1)$th derivative which is equal to

$$\lambda^{(k-1)}(t_1)\phi(t_1) + (k-1)\lambda^{(k-2)}(t_1)\phi'(t_1) + \cdots + \lambda(t_1)\phi^{(k-1)}(t_1). \tag{18}$$

Now $\phi^{(k-1)}(t_1)$, having a continuous derivative $\phi^{(k)}(t_1)$, is *a fortiori* of limited variation, and the same is true of the earlier derivatives of ϕ and of ϕ itself. Similarly, the derivatives of λ of orders from 0 to $k-2$ are of limited variation as well as the $(k-1)$th. Consequently the property of limited variation is possessed by the whole expression (18). From the corresponding expansions of the earlier derivatives of the product $\lambda\phi$ it is seen that each vanishes at both ends of the interval $(0, \pi)$. It is recognized thus that $\lambda\phi$ satisfies the conditions imposed on ϕ in Lemma I, except that k in the lemma is to be replaced by $k-1$. By that lemma, the integral in (17), without the factor $1/\rho_n$, is $O(1/n^k)$, and when divided by ρ_n becomes $O(1/n^{k+1})$.

Since $\lambda(t_1)\phi''(t_1)$ satisfies the conditions of Lemma I with k replaced by $k-2$, we have

$$\frac{1}{\rho_n^3}\int_0^\pi \lambda(t_1)\phi''(t_1)\cos\rho_n t_1\,dt_1 = O\left(\frac{1}{n^{k+2}}\right).$$

Each of the remaining terms can be disposed of in a similar manner; at each step, from this point on, the number of derivatives of limited variation known to be possessed by the integrand is diminished by 2, and this loss is compensated by the presence of a higher power of $1/\rho_n$ before the integral.

It appears then that (14) is true. The proof of (15) obviously follows the same lines.

Let us look again for a moment at the special case that $\phi^{(k)}(0) = 0$. If v still represents the total variation of $\phi^{(k)}(x)$ in $(0, \pi)$,

$$|\phi^{(k)}(x)| \leqq v$$

throughout the interval. Hence the total variation of $\phi^{(k-1)}(x)$ in the interval can not exceed πv. The absolute value of $\phi^{(k-1)}(x)$ can not exceed the same quantity, since $\phi^{(k-1)}(0) = 0$. In the same way the total variation of $\phi^{(k-2)}(x)$ and the maximum of its absolute value are less than or equal to $\pi^2 v$, and so on.

It is readily found that the total variation of $\lambda\phi$, for example, does not exceed v multiplied by a quantity independent of ϕ. We can write

$$\lambda\phi = (\lambda_1 - \lambda_2)(\phi_1 - \phi_2) = (\lambda_1\phi_1 + \lambda_2\phi_2) - (\lambda_1\phi_2 + \lambda_2\phi_1),$$

where λ_1, λ_2, ϕ_1, and ϕ_2 are monotone increasing and continuous, and the last two vanish at the point 0. It can be assumed also that λ_1 and λ_2 are positive or zero throughout the interval. Then if V is the total variation of $\lambda\phi$,

$$V \leqq \lambda_1(b)\phi_1(b) + \lambda_2(b)\phi_2(b) + \lambda_1(b)\phi_2(b) + \lambda_2(b)\phi_1(b)$$
$$\leqq 2[\lambda_1(b) + \lambda_2(b)]\,\pi^k v.$$

By similar reasoning, it is found that the $(k-1)$th derivative of $\lambda\phi$, expanded as in (18), and the $(k-2)$th derivative of $\lambda\phi''$, etc., have the same property, namely, that the total variation of each is inferior to v multiplied by a quantity independent of ϕ. Applying this fact in the preceding proof of Lemma II, we conclude that when $\phi^{(k)}(0) = 0$ the statement of the lemma can be made more precise by saying that the absolute value of each of the two integrals concerned remains inferior to cv/n^{k+1}.

LEMMA III. *If ϕ and λ satisfy the conditions of Lemma II, then**

$$(19)\quad \begin{aligned} &\int_0^\pi \phi(x)\int_0^x \lambda(t_1)\sin\rho_n(x-t_1)\int_0^{t_1}\lambda(t_2)\sin\rho_n(t_1-t_2)\int_0^{t_2}\cdots \\ &\qquad\int_0^{t_{s-1}}\lambda(t_s)\cos\rho_n t_s \sin\rho_n(t_{s-1}-t_s)\,dt_s\,dt_{s-1}\cdots dt_1\,dx = O\left(\frac{1}{n^{k+1}}\right), \end{aligned}$$

and the same is true if $\cos\rho_n t_s$ *in the innermost integral is replaced by* $\sin\rho_n t_s$.

The proof is obtained by induction. The statement is already known to be true when $s = 1$, as it reduces then to Lemma II, the variable x taking the place of a variable t_0. We shall assume that the conclusion holds if s is replaced by $s-1$, and show then that it holds as stated. The steps in the passage from $s-1$ to s correspond exactly to those carried out in the proof of Lemma II. We begin by inverting the order of integration, confining our attention to the integral with $\cos\rho_n t_s$. The given expression is equal to

$$\begin{aligned} &\int_0^\pi \lambda(t_s)\cos\rho_n t_s\int_{t_s}^\pi \lambda(t_{s-1})\sin\rho_n(t_{s-1}-t_s)\int_{t_{s-1}}^\pi\cdots \\ &\qquad\int_{t_2}^\pi \lambda(t_1)\sin\rho_n(t_1-t_2)\int_{t_1}^\pi \phi(x)\sin\rho_n(x-t_1)\,dx\,dt_1\cdots dt_{s-1}\,dt_s. \end{aligned}$$

The innermost integral here is precisely that which appears in (16). By the aid of that formula, we express the $(s+1)$-fold integral as the sum of $\gamma+1$ terms, of which the last is immediately seen to be $O(1/n^{k+1})$, while the others

* In this multiple integral, the notation for which is suggestive rather than complete, $s-3$ more factors of the form $\lambda(t_i)\sin\rho_n(t_{i-1}-t_i)$ are to be understood. It will be seen a few lines below how an integral of this form is obtained as a result of successive substitutions in (5).

are shown to be $O(1/n^{k+1})$, or even $O(1/n^{k+2})$, by means of the assumed lemma for s-fold integrals, with ϕ replaced successively by $\lambda\phi$, $\lambda\phi''$, $\cdots$, $\lambda\phi^{(2\gamma-2)}$. Thus the relation (19) is established, and clearly remains true if $\sin \rho_n t_s$ is written instead of $\cos \rho_n t_s$.

In the special case that $\phi^{(k)}(0) = 0$, it is readily shown by the use of the corresponding refinement of Lemma II that each of the two integrals in the present lemma remains inferior in absolute value to cv/n^{k+1}.

We are now in a position to deal with the integral (8). If, on the right-hand side of (5), the function $U_n(t)$ under the integral sign is expressed at length by means of (5) itself, the following more extended formula results:

$$
\begin{aligned}
U_n(x) = \cos \rho_n x + \frac{h' \sin \rho_n x}{\rho_n} + \frac{1}{\rho_n}\int_0^x \lambda(t_1) \cos \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1 \\
+ \frac{h'}{\rho_n^2}\int_0^x \lambda(t_1) \sin \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1 \\
+ \frac{1}{\rho_n^2}\int_0^x \lambda(t_1) \sin \rho_n (x - t_1) \int_0^{t_1} \lambda(t_2)\, U_n(t_2) \sin \rho_n (t_1 - t_2)\, dt_2\, dt_1 .
\end{aligned}
\tag{20}
$$

There is still an integral on the right-hand side containing U_n under the integral sign, and the step just taken can be repeated. This process is to be continued until an expression is obtained in which the integral involving U_n is preceded by a factor $1/\rho_n^{k+1}$, where k is an appropriate integer. Let this expression be multiplied by $\phi(x)$ and integrated from 0 to π with regard to x. The last of the resulting terms will be $O(1/n^{k+1})$, inasmuch as the integral, without the factor $1/\rho_n^{k+1}$, remains finite for all values of n. The integrals which appear in the earlier terms are explicit expressions of the types considered in Lemmas I, II, and III. If ϕ and λ are restricted as in those lemmas, each integral, even without the power of $1/\rho_n$ which stands before it, will be $O(1/n^{k+1})$. If λ is supposed provided with only $k - 2$ continuous derivatives of limited variation, instead of $k - 1$, so that k is to be replaced by $k - 1$ in applying Lemmas II and III, each term will still be $O(1/n^{k+1})$ after the factors $1/\rho_n$ have been multiplied in. The conclusion may be stated thus:

LEMMA IV. *If ϕ satisfies the conditions of Lemma I, and if λ has a continuous $(k - 2)$th derivative of limited variation in $0 \leqq x \leqq \pi$, then*

$$\int_0^\pi \phi(x)\, U_n(x)\, dx = O\left(\frac{1}{n^{k+1}}\right). \tag{21}$$

If, in addition, λ has a continuous $(k - 1)$th derivative of limited variation in the interval,

$$\int_0^\pi \phi(x)\, U_n(x)\, dx = \int_0^\pi \phi(x) \cos \rho_n x\, dx + O\left(\frac{1}{n^{k+2}}\right). \tag{22}$$

If $\phi^{(k)}(0) = 0$, the constant multiplier implied in the O-notation can be taken as v, the total variation of $\phi^{(k)}(x)$, multiplied by a quantity independent* of ϕ.

Let the case $k = 1$ be brought up again for special consideration at this point. The hypothesis for the first part of Lemma IV takes the form that ϕ has a continuous first derivative of limited variation and that

$$\phi(0) = \phi(\pi) = 0,$$

while nothing more than continuity is required of λ. The point to be made is that in this case the special restriction on ϕ at the ends of the interval is unnecessary. Let $U_n(x)$ be expanded to five terms as in (20), multiplied by $\phi(x)$, and integrated. It has already been pointed out, in connection with the proof of Lemma I, that

$$\int_0^{\pi} \phi(x) \cos \rho_n x \, dx = O\left(\frac{1}{n^2}\right),$$

whether ϕ vanishes at the ends of the interval or not. As for the other terms, the application of Lemma II with $k = 0$ gives all that is required.†

The discussion so far has related to the numerator of the expression (7) for α_n. For the denominator we need only the simple relation given by Liouville:‡

$$\int_0^{\pi} [U_n(x)]^2 \, dx = \frac{\pi}{2} + O\left(\frac{1}{n}\right). \tag{23}$$

The proof is simply this: In consequence of (5),

$$\int_0^{\pi} [U_n(x)]^2 \, dx = \int_0^{\pi} \cos^2 \rho_n x \, dx + O\left(\frac{1}{n}\right),$$

and

$$\int_0^{\pi} \cos^2 \rho_n x \, dx = \frac{\pi}{2} + \frac{\sin 2\rho_n \pi}{4\rho_n} = \frac{\pi}{2} + O\left(\frac{1}{n}\right).$$

We have the materials now for one theorem on the degree of convergence

* It is to be noticed that the formulas on which all our demonstrations depend hold only for values of *n from a certain point on,* but it is readily seen that the conclusions are correct for all positive values of n. This remark is not trivial in the case of such a proposition as the one to which this note is appended. It is asserted, for example, that

$$\int_0^{\pi} \phi(x) \, U_1(x) \, dx$$

does not exceed a constant multiple of v; this is true with the hypotheses that are stated, since the maximum of $|\phi|$ does not exceed a constant multiple of v, but would not generally be true otherwise.

† If this lemma and (20) had not already been written out at length, it would have been still simpler to operate with the three-term expression (5) directly.

‡ Loc. cit., p. 430.

of the series (6). If ϕ and λ satisfy the hypotheses for the first part of Lemma IV, it follows from (21) and (23) that

$$\alpha_n = O\left(\frac{1}{n^{k+1}}\right),$$

and as the maximum of $|U_n(x)|$ remains finite when n becomes infinite,

$$|\alpha_n U_n(x)| \leqq \frac{\beta}{n^{k+1}},$$

where β is independent of n and x. It follows that the series $\sum \alpha_n U_n(x)$ converges uniformly,* in which case, as has already been stated, it must converge to the value $\phi(x)$. If we set

$$\sigma_n(x) = \sum_{\nu=0}^{n} \alpha_\nu U_\nu(x),$$

then

$$|\phi(x) - \sigma_n(x)| = \left|\sum_{\nu=n+1}^{\infty} \alpha_\nu U_\nu(x)\right| \leqq \sum_{\nu=n+1}^{\infty} \frac{\beta}{\nu^{k+1}} < \beta \int_n^\infty \frac{dt}{t^{k+1}} = \frac{\beta}{kn^k}.$$

That is:

THEOREM I. *If $\phi(x)$ has a continuous kth derivative of limited variation in the interval $0 \leqq x \leqq \pi$, while ϕ itself and its first $k-1$ derivatives vanish for $x = 0$ and for $x = \pi$, and if, furthermore, the function $\lambda(x)$ which appears as a coefficient in the differential equation has a continuous $(k-2)$th derivative of limited variation in $0 \leqq x \leqq \pi$, then*

$$\phi(x) = \sigma_n(x) + O\left(\frac{1}{n^k}\right)$$

uniformly throughout the interval.

Whenever the O-notation is used in the present paper in a relation involving a function of x, it will be understood, without being repeated on every occasion, that the relation holds uniformly, that is, that the implicit constant multiplier is independent of x as well as of n. In the relation just written down it depends on k, λ, h', H', and ϕ. In the case that $\phi^{(k)}(0) = 0$ its dependence on ϕ is completely characterized by saying that it is proportional to the total variation of $\phi^{(k)}$.

When $k = 1$, the theorem is true without the restriction that

$$\phi(0) = \phi(\pi) = 0,$$

in consequence of the facts previously established for this special case. That the restrictions at the ends of the interval are not superfluous in general is shown by the simplest examples. If

$$\lambda(x) = 0, \qquad h' = H' = 0, \qquad \phi(x) = \sin x,$$

* Of course this is assured at the outset by well-known general theorems, but it appears incidentally here.

for instance, so that the series is the cosine-series for $\sin x$, the hypotheses of Theorem I, except those relating to the end-points, are satisfied for any value of k, but the remainder in the series

$$\sin x = \frac{4}{\pi}\left[1 - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \cdots - \frac{\cos 2nx}{4n^2 - 1} - \cdots\right]$$

at the point $x = \frac{1}{2}\pi$ does not approach zero faster than $1/n^2$.

A further theorem can be obtained by operating with (22). Let the equation (12) be recalled, and compared with the following, which is obtained in the same way:

$$\int_0^{\pi} \phi(x) \cos nx \, dx = \frac{1}{n^k} \int_0^{\pi} \phi^{(k)}(x) \cos\left(nx + \frac{k\pi}{2}\right) dx.$$

By (3) and (4) we may write

$$\frac{1}{\rho_n^k} = \frac{1}{n^k}\left(\frac{n}{\rho_n}\right)^k = \frac{1}{n^k}\left(1 - \frac{\epsilon_n}{\rho_n}\right)^k = \frac{1}{n^k} + O\left(\frac{1}{n^{k+2}}\right).$$

On the other hand,

$$\cos\left(\rho_n x + \frac{k\pi}{2}\right) = \cos\left(nx + \frac{k\pi}{2}\right) - (1 - \cos \epsilon_n x)\cos\left(nx + \frac{k\pi}{2}\right)$$

$$- \sin \epsilon_n x \sin\left(nx + \frac{k\pi}{2}\right)$$

$$= \cos\left(nx + \frac{k\pi}{2}\right) - \sin \epsilon_n x \sin\left(nx + \frac{k\pi}{2}\right) + O\left(\frac{1}{n^2}\right).$$

Now $\sin \epsilon_n x$ is a positive monotone increasing function in $(0, \pi)$, as soon as n is sufficiently large, and so, if we write

$$\phi^{(k)}(x) = \phi_1(x) - \phi_2(x),$$

as in (13), we shall have

$$\int_0^{\pi} \phi^{(k)}(x) \sin \epsilon_n x \sin\left(nx + \frac{k\pi}{2}\right) dx$$

$$= \phi_1(\pi) \sin \epsilon_n \pi \int_{\xi}^{\pi} \sin\left(nx + \frac{k\pi}{2}\right) dx - \phi_2(\pi) \sin \epsilon_n \pi \int_{\eta}^{\pi} \sin\left(nx + \frac{k\pi}{2}\right) dx$$

$$= O\left(\frac{1}{n^2}\right),$$

since $\sin \epsilon_n \pi$ and both the integrals are separately $O(1/n)$. By combination of these relations,

$$\frac{1}{\rho_n^k}\int_0^\pi \phi^{(k)\prime}(x)\cos\left(\rho_n x+\frac{k\pi}{2}\right)dx$$

$$=\left[\frac{1}{n^k}+O\left(\frac{1}{n^{k+2}}\right)\right]\left[\int_0^\pi \phi^{(k)}(x)\cos\left(nx+\frac{k\pi}{2}\right)dx+O\left(\frac{1}{n^2}\right)\right]$$

$$=\frac{1}{n^k}\int_0^\pi \phi^{(k)}(x)\cos\left(nx+\frac{k\pi}{2}\right)dx+O\left(\frac{1}{n^{k+2}}\right).$$

That is,

$$\int_0^\pi \phi(x)\cos\rho_n x\,dx=\int_0^\pi \phi(x)\cos nx\,dx+O\left(\frac{1}{n^{k+2}}\right);$$

the relation (22) remains true if $\cos\rho_n x$ is replaced by $\cos nx$. The integral on the right is itself $O(1/n^{k+1})$. From (23),

$$\frac{1}{\int_0^\pi [U_n(x)]^2\,dx}=\frac{2}{\pi}+O\left(\frac{1}{n}\right).$$

Hence, by multiplication,

$$\alpha_n=\frac{2}{\pi}\int_0^\pi \phi(x)\cos nx\,dx+O\left(\frac{1}{n^{k+2}}\right).$$

To go one step further,

$$U_n(x)=\cos\rho_n x+O\left(\frac{1}{n}\right)=\cos nx+O\left(\frac{1}{n}\right),$$

and so

$$\alpha_n U_n(x)=\frac{2}{\pi}\cos nx\int_0^\pi \phi(t)\cos nt\,dt+O\left(\frac{1}{n^{k+2}}\right). \tag{24}$$

Up to this point the original hypotheses about the function ϕ have been retained. It is for a somewhat different class of functions that the last relations are to be used. Suppose that $\phi(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition,

$$|\phi^{(k-1)}(x_2)-\phi^{(k-1)}(x_1)|\leqq\mu|x_2-x_1|, \tag{25}$$

where x_1 and x_2 are any two values in the closed interval $(0,\pi)$ and μ is a constant. This derivative is *a fortiori* continuous and of limited variation, and if we suppose that ϕ and its first $k-2$ derivatives vanish at 0 and π, and that λ has a continuous derivative of order $k-2$ with limited variation, the hypotheses for (24) will be fulfilled with k replaced by $k-1$.

The function $\phi(x)$ has been defined up to the present only in the interval from 0 to π. Let this definition be extended by setting $\phi(x)=\phi(-x)$ for $-\pi\leqq x\leqq 0$, and then making $\phi(x+2\pi)=\phi(x)$ for all real values of x. The periodic function so defined will have a $(k-1)$th derivative satisfying a Lipschitz condition everywhere* if k is odd, and this will be true for even values of k as well if the hypotheses previously made are supplemented by requiring that $\phi^{(k-1)}(0) = \phi^{(k-1)}(\pi) = 0$.

The Fourier's series for $\phi(x)$ will involve only cosine-terms, since ϕ is even, and will have for its general term precisely the expression on the right-hand side of (24), with the O-term omitted. Let this general term of the Fourier's series be denoted by $a_n \cos nx$, the sum of the first $n + 1$ terms by $s_n(x)$. It is known that†

$$\phi(x) - s_n(x) = \sum_{\nu=n+1}^{\infty} a_\nu \cos \nu x = O\left(\frac{\log n}{n^k}\right). \tag{26}$$

Remembering that k in (24) is to be replaced by $k - 1$, we have

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq \frac{\beta}{n^{k+1}}, \tag{27}$$

where β is independent of n and x. It is understood that‡ $k \geqq 1$; it appears from the relations just written down that the Sturm-Liouville series $\sum \alpha_n U_n(x)$ converges uniformly, and that

$$\begin{aligned}|\phi(x) - \sigma_n(x)| &\leqq |\phi(x) - s_n(x)| + |[\phi(x) - \sigma_n(x)] - [\phi(x) - s_n(x)]| \\ &= |\phi(x) - s_n(x)| + \left|\sum_{\nu=n+1}^{\infty} [\alpha_\nu U_\nu(x) - a_\nu \cos \nu x]\right| \\ &\leqq |\phi(x) - s_n(x)| + \sum_{\nu=n+1}^{\infty} \frac{\beta}{\nu^{k+1}} \\ &= O\left(\frac{\log n}{n^k}\right) + O\left(\frac{1}{n^k}\right) = O\left(\frac{\log n}{n^k}\right).\end{aligned}$$

This may be formulated as follows:

THEOREM II. *If $\phi(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition throughout the interval $0 \leqq x \leqq \pi$, while ϕ itself and its first $k - 2$ derivatives, and, in case k is even, the $(k-1)$th derivative also, vanish for*

* It is seen at once that ϕ itself is continuous, and readily shown that the successive derivatives through the $(k-2)$th exist and are continuous, even at the points $0, \pi, 2\pi \cdots$. The $(k-2)$th derivative will be an even or an odd function according as k is even or odd. In the latter case its right-hand and left-hand derivatives at the point 0 will be equal; in the former they will be the negatives of each other, and so equal only if they vanish. Similar reasoning applies to the point π. The others need not be considered separately, because of the periodicity of ϕ. The proof that the Lipschitz condition is satisfied offers no difficulty.

† For the case $k = 1$, see Lebesgue, loc. cit., p. 201; for the general case, D. Jackson, loc. cit., also D. Jackson, these Transactions, vol. 14 (1913), pp. 343–364, Theorem X. The latter paper will be referred to as B.

‡ The meaning of the hypothesis in the case $k = 1$ would be, of course, that ϕ itself satisfies a Lipschitz condition.

$x = 0$ and for $x = \pi$, and if $\lambda(x)$ has a continuous $(k-2)$th derivative of limited variation for $0 \leqq x \leqq \pi$, then

$$\phi(x) = \sigma_n(x) + O\left(\frac{\log n}{n^k}\right)$$

uniformly throughout the interval.

If k is even, so that it is assumed that $\phi^{(k-1)}(0) = 0$, or if k is odd and this assumption is added to those already made, it will be seen, on following through the work with this in view, that the remainder does not exceed $c\mu(\log n)/n^k$, where c is independent of ϕ, and μ is the coefficient in the Lipschitz condition which $\phi^{(k-1)}(x)$ satisfies,* and it is assumed that $n \geqq 2$. It is essential for this purpose to note from the papers referred to for the Fourier's series that the constant factor implicit in the O-symbol in (26) may be taken as μ multiplied by a quantity independent of ϕ.

The suggested refinement of Theorem II may be dismissed with these few lines, as far as general values of k are concerned. In the simplest case, $k = 1$, the theorem is needed for an application in its more precise form, and it will be well to state this simple result separately and to give the details of the proof from the beginning.†

THEOREM II*a*. *If $\phi(x)$ satisfies the Lipschitz condition*

$$|\phi(x_2) - \phi(x_1)| \leqq \mu |x_2 - x_1| \tag{28}$$

throughout the interval $0 \leqq x \leqq \pi$, and if $\phi(0) = 0$, then, in the whole interval,

$$|\phi(x) - \sigma_n(x)| \leqq \frac{c\mu \log n}{n}, \qquad n \geqq 2,$$

where c is independent of x, n, and ϕ.

The restriction on $\lambda(x)$ is merely that of continuity.

In accordance with the hypothesis, $|\phi(x)| \leqq \pi\mu$, and $\phi(x)$ can be written in the form

$$\phi(x) = \phi_1(x) - \phi_2(x), \tag{29}$$

where ϕ_1 and ϕ_2 are continuous and monotone increasing, and

$$\phi_1(0) = \phi_2(0) = 0, \qquad \phi_1(\pi) \leqq \pi\mu, \qquad \phi_2(\pi) \leqq \pi\mu.$$

The expression (5) for $U_n(x)$ will be sufficient for our purpose. Let the second term be multiplied by $\phi(x)$ and integrated; we find

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* It is obvious that the total variation of $\phi^{(k-1)}(x)$ in $(0, \pi)$ is at most $\pi\mu$.

† This will involve some repetition of what has gone before, which is perhaps compensated by the gain in clearness.

$$\left|\int_0^\pi \phi(x)\sin\rho_n x\,dx\right| = \left|\phi_1(\pi)\int_\xi^\pi \sin\rho_n x\,dx - \phi_2(\pi)\int_\eta^\pi \sin\rho_n x\,dx\right|$$
$$\leqq \frac{4\pi\mu}{\rho_n} \leqq \frac{c\mu}{n},$$

and hence

$$\left|\frac{h'}{\rho_n}\int_0^\pi \phi(x)\sin\rho_n x\,dx\right| \leqq \frac{c\mu}{n^2};$$

as has already been explained, the letter c will be used to represent a number independent of n and ϕ, and, in a relation where x occurs, independent also of x, but will stand for different numbers of this sort in different lines, and, on occasion, even in the same line. Consider the third term:

$$\int_0^\pi \phi(x)\int_0^x \lambda(t)\,U_n(t)\sin\rho_n(x-t)\,dt\,dx$$
$$= \int_0^\pi \lambda(t)\,U_n(t)\cos\rho_n t\int_t^\pi \phi(x)\sin\rho_n x\,dx\,dt$$
$$- \int_0^\pi \lambda(t)\,U_n(t)\sin\rho_n t\int_t^\pi \phi(x)\cos\rho_n x\,dx\,dt,$$

and as each of the integrals extended from t to π is seen by the use of (29) and the second law of the mean to be in absolute value not greater than $c\mu/n$, it follows that the whole expression satisfies an inequality of the same form. Consequently

$$\left|\frac{1}{\rho_n}\int_0^\pi \phi(x)\int_0^x \lambda(t)\,U_n(t)\sin\rho_n(x-t)\,dt\,dx\right| \leqq \frac{c\mu}{n^2}.$$

Return now to the first term of (5); here it is to be pointed out that

$$\left|\int_0^\pi \phi(x)\cos\rho_n x\,dx - \int_0^\pi \phi(x)\cos nx\,dx\right|$$
$$\leqq \left|\int_0^\pi (\cos\epsilon_n x - 1)\,\phi(x)\cos nx\,dx\right| + \left|\int_0^\pi \phi(x)\sin\epsilon_n x\sin nx\,dx\right|$$
$$\leqq \frac{c\mu}{n^2} + \left|\phi_1(\pi)\sin\epsilon_n\pi\int_\xi^\pi \sin nx\,dx - \phi_2(\pi)\sin\epsilon_n\pi\int_\eta^\pi \sin nx\,dx\right|$$
$$\leqq \frac{c\mu}{n^2},$$

the last inequality but one being obtained as soon as n is so large that* $\epsilon_n \leqq \frac{1}{2}$ and $\sin\epsilon_n x$ is monotone increasing in $(0, \pi)$. To sum up the inequalities obtained thus far,

$$(30)\qquad \int_0^\pi \phi(x)\,U_n(x)\,dx = \int_0^\pi \phi(x)\cos nx\,dx + r_1(n, x),$$

* The point where this begins to be true is of course independent of ϕ.

where the remainder r_1 is such that

$$|r_1(n, x)| \leqq \frac{c\mu}{n^2}.$$

In consequence of (23), which was proved in four lines, we may write

$$\frac{1}{\int_0^\pi [U_n(x)]^2\,dx} = \frac{2}{\pi} + r_2(n), \tag{31}$$

where

$$|r_2(n)| \leqq \frac{c}{n}.$$

Furthermore,

$$U_n(x) = \cos \rho_n x + r_3(n, x) = \cos nx + r_4(n, x), \tag{32}$$

where

$$|r_3(n, x)| \leqq \frac{c}{n}, \qquad |r_4(n, x)| \leqq \frac{c}{n}.$$

As the second law of the mean shows that

$$\left| \int_0^\pi \phi(x) \cos nx\,dx \right| \leqq \frac{c\mu}{n},$$

multiplication of (30), (31), and (32) gives

$$\alpha_n U_n(x) = \frac{2}{\pi} \cos nx \int_0^\pi \phi(t) \cos nt\,dt + r_5(n, x), \tag{33}$$

where

$$|r_5(n, x)| \leqq \frac{c\mu}{n^2}. \tag{34}$$

Let $\phi(x)$ be defined outside of $(0, \pi)$ so as to be an even function of period 2π for all real values of x. This function will satisfy (28) everywhere. If $a_n \cos nx$ is the general term of its Fourier's series, which contains no sine-terms, and

$$s_n(x) = \sum_{\nu=0}^{n} a_n \cos nx,$$

then*

$$|\phi(x) - s_n(x)| \leqq \frac{c\mu \log n}{n}, \qquad n \geqq 2.$$

The relation (33), with (34), states that

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq \frac{c\mu}{n^2}.$$

* Lebesgue, loc. cit., p. 201; D. Jackson, in the paper A, Theorem V.

This has been established only for values of n from a certain point on, this point being unspecified, but independent of ϕ. On taking account of the latter circumstance, it is readily seen that the relation holds also for the positive values of n previously neglected. For the inequalities

$$|\alpha_n U_n(x)| \leqq c\mu, \qquad |a_n \cos nx| \leqq c\mu,$$

are seen at once to hold for all these values of n, since $|\phi(x)| \leqq \pi\mu$, and $U_n(x)$, whether represented by (5) or not, is continuous and not identically zero; it follows that

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq c\mu,$$

and it is immaterial whether we write $c\mu$ or $c\mu/n^2$, since only a finite number of values of n are concerned. Consequently, for $n \geqq 2$,

$$\begin{aligned} |\phi(x) - \sigma_n(x)| &\leqq |\phi(x) - s_n(x)| + |[\phi(x) - \sigma_n(x)] - [\phi(x) - s_n(x)]| \\ &= |\phi(x) - s_n(x)| + \left| \sum_{\nu=n+1}^{\infty} [\alpha_\nu U_\nu(x) - a_\nu \cos \nu x] \right| \\ &\leqq |\phi(x) - s_n(x)| + \sum_{\nu=n+1}^{\infty} \frac{c\mu}{n^2} \\ &\leqq \frac{c\mu \log n}{n} + \frac{c\mu}{n} \leqq \frac{c\mu \log n}{n}. \end{aligned}$$

Now let $\phi(x)$ be any function whatever that is continuous in $0 \leqq x \leqq \pi$ and vanishes for $x = 0$. Represent by δ any positive quantity, and let $\omega(\delta)$ denote the maximum of $|\phi(x_2) - \phi(x_1)|$ for values of x_1 and x_2 in the interval subject to the restriction that $|x_2 - x_1| \leqq \delta$. It is obvious at once that $\omega(\delta)$ is a monotone increasing function of δ, and that

$$\lim_{\delta=0} \omega(\delta) = 0.$$

It may be assumed further that $\omega(\delta)/\delta$ does not approach zero as δ approaches zero, otherwise ϕ would have a vanishing derivative throughout the interval and so would vanish identically itself. The relation just obtained is to be used to establish the following proposition:

THEOREM III. *If* $\phi(x)$ *is continuous in* $0 \leqq x \leqq \pi$ *and*

$$|\phi(x_2) - \phi(x_1)| \leqq \omega(\delta) \tag{35}$$

whenever

$$|x_2 - x_1| \leqq \delta,$$

and if $\phi(0) = 0$, *then*

$$|\phi(x) - \sigma_n(x)| \leqq c\omega\left(\frac{\pi}{n}\right) \log n, \qquad n \geqq 2.$$

The proof, after Theorem II*a* has been established, is that given by Lebesgue* for the corresponding property of Fourier's series. We begin by deriving the following lemma:

LEMMA V. *If*

$$|\phi(x)| \leqq \epsilon$$

throughout the interval $0 \leqq x \leqq \pi$, *then*

$$|\phi(x) - \sigma_n(x)| \leqq c\epsilon \log n, \qquad n \geqq 2.$$

It is obviously sufficient to show that $|\sigma_n(x)|$ itself can not exceed $c\epsilon \log n$. Now it follows at once from (5), with (3) and (4), that

$$\left| \int_0^\pi \phi(x) U_n(x) dx - \int_0^\pi \phi(x) \cos \rho_n x \, dx \right| \leqq \frac{c\epsilon}{n},$$

or, since $|\cos \rho_n x - \cos nx| \leqq c/n$, that

$$\left| \int_0^\pi \phi(x) U_n(x) dx - \int_0^\pi \phi(x) \cos nx \, dx \right| \leqq \frac{c\epsilon}{n}.$$

From this and (31) and (32),

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq \frac{c\epsilon}{n}.$$

Therefore

$$|\sigma_n(x) - s_n(x)| \leqq \sum_{\nu=1}^{n} \frac{c\epsilon}{\nu} \leqq c\epsilon \log n,$$

and as it is well known† that

$$|s_n(x)| \leqq c\epsilon \log n,$$

the lemma follows at once.

Now let a function $\psi(x)$ be defined as equal to $\phi(x)$ at the points $x = i\pi/n$, $i = 0, 1, \cdots, n$, and linear between these points. Because of (35),

$$(36) \qquad |\phi(x) - \psi(x)| \leqq 2\omega\left(\frac{\pi}{n}\right)$$

throughout the interval. Furthermore, $\psi(x)$ satisfies (28) with

$$(37) \qquad \mu = \frac{\omega(\pi/n)}{\pi/n}.$$

Let us denote by $\sigma_n(x)$ the sum of the first $n + 1$ terms of the Sturm-Liouville series for $\phi(x)$, and by $\sigma_{1n}(x)$ and $\sigma_{2n}(x)$ the corresponding sums formed for $\psi(x)$ and $\phi(x) - \psi(x)$ respectively. Of course

$$\sigma_n(x) = \sigma_{1n}(x) + \sigma_{2n}(x).$$

* Loc. cit., pp. 201–202.

† See, e. g., Lebesgue, loc. cit., pp. 196–197.

By (37) and Theorem II*a*,

$$|\psi(x) - \sigma_{1n}(x)| \leqq \frac{c \log n}{n} \cdot \frac{\omega(\pi/n)}{\pi/n} = c\omega\left(\frac{\pi}{n}\right) \log n.$$

By (36) and Lemma V,

$$|\sigma_{2n}(x)| \leqq c\omega\left(\frac{\pi}{n}\right) \log n.$$

As it follows from (36) *a fortiori* that

$$|\phi(x) - \psi(x)| \leqq c\omega\left(\frac{\pi}{n}\right) \log n,$$

the identity

$$\phi(x) - \sigma_n(x) = \phi(x) - \psi(x) + \psi(x) - \sigma_{1n}(x) - \sigma_{2n}(x)$$

gives the demonstration of Theorem III at once.

If $\phi(x)$ is continuous, but $\phi(0) \neq 0$, let

$$\phi(x) = \phi(0) + \chi(x).$$

Theorem I is applicable to the Sturm-Liouville series for the constant $\phi(0)$, with $k = 1$; for of course a constant has a first derivative of limited variation, and it was pointed out in connection with that theorem that for $k = 1$ the function developed need not vanish at the ends of the interval. Hence the remainder after n terms of the series is $O(1/n)$. On the other hand, Theorem III may be applied to the function $\chi(x)$. If $\omega(\delta)$, formed for this function, is such that the ratio of δ to $\omega(\delta)$ remains finite as δ approaches zero,* then

$$\frac{1}{n} = O\left[\omega\left(\frac{\pi}{n}\right)\right] = O\left[\omega\left(\frac{\pi}{n}\right) \log n\right],$$

and $\phi(x)$ still has the property that

$$\phi(x) = \sigma_n(x) + O\left[\omega\left(\frac{\pi}{n}\right) \log n\right],$$

but it is no longer true that the constant multiplier in the O-symbol is independent of ϕ.

3. Summation of the Series

The theorem expressed in (26), concerning the degree of convergence of the Fourier's series for a function $\phi(x)$ of period 2π having a $(k-1)$th derivative that satisfies a Lipschitz condition, was proved by an indirect method. It was obtained as a consequence of the two following propositions:

(*a*) If $\phi(x)$ is a function of the character described, it is possible to define for each positive integral value of n a finite trigonometric sum of the nth order

* This assumption goes somewhat beyond the earlier one, that $\omega(\delta)/\delta$ shall not approach zero, that is, that the ratio of δ to $\omega(\delta)$ shall not become infinite.

at most which represents $\phi(x)$ approximately with a maximum error not exceeding a constant multiple of* $1/n^k$.

(*b*) If $\phi(x)$ is any function† of period 2π which can be approximately represented by a finite trigonometric sum of the nth or lower order with an error nowhere exceeding ϵ, and if $s_n(x)$ is the corresponding partial sum of the Fourier's series for $\phi(x)$, then, for $n \geqq 2$,

$$|\phi(x) - s_n(x)| \leqq K\epsilon \log n,$$

where K is an absolute constant.‡

It may not be unprofitable to observe that each of these propositions, which are interesting for their own sake, has its analogue in the theory of the Sturm-Liouville series. For (*b*) this analogue reads as follows:

If $\phi(x)$ can be approximately represented throughout the closed interval $(0, \pi)$ by a linear combination $\Sigma_n(x)$ of the characteristic functions $U_0(x)$, $U_1(x)$, $\cdots$, $U_n(x)$, with constant coefficients, with an error nowhere exceeding ϵ, and if $\sigma_n(x)$ denotes the corresponding partial sum of the Sturm-Liouville series for $\phi(x)$, then, for $n \geqq 2$,

$$|\phi(x) - \sigma_n(x)| \leqq c\epsilon \log n$$

throughout the interval.

Here, as everywhere in this paper, c denotes a constant independent of ϕ, but depending conceivably on $\lambda(x)$ and on the coefficients in the boundary conditions.

The truth of the assertion is an immediate consequence of Lemma V. The partial sum of the Sturm-Liouville series for $\phi(x)$ is obtained by adding those for $\Sigma_n(x)$ and for $\phi(x) - \Sigma_n(x)$; the partial sum for $\Sigma_n(x)$ is $\Sigma_n(x)$ itself, and an error can arise only in the partial sum for $\phi(x) - \Sigma_n(x)$, to which Lemma V applies.

The generalization of (*a*) is somewhat less trivial. Let us recall the method by which that theorem itself was proved in the paper A.§

If k is the integer which appears in the statement of the theorem, and n is any positive integer, let κ be the smallest integer for which $2\kappa - k > 1$, and m the largest integer for which $\kappa(m-1)$ does not exceed n, and let

$$S_n(x) = h_m \int_{-\pi/2}^{\pi/2} [(-1)^{k+1}\phi(x+2ku) + (-1)^k k\phi(x+2(k-1)u) + \cdots + k\phi(x+2u)] \left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du, \tag{38}$$

* D. Jackson, Dissertation, Göttingen 1911, Theorem VII; A, Theorem III; B, Theorem III.

† Integrable in the sense of Lebesgue.

‡ Lebesgue, loc. cit., p. 201; also, earlier, Lebesgue, *Annales de la Faculté des Sciences de l'Université de Toulouse pour les sciences mathématiques et les sciences physiques*, series 3, vol. 1 (1909), pp. 25–117; pp. 116–117.

§ The neater proof given in B seems less convenient for the present purpose.

where the numerical coefficients in the integrand are the binomial coefficients corresponding to the exponent k, the last being omitted, and h_m is a constant defined by the equation

$$\frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} \left[\frac{\sin\, mu}{m \sin u} \right]^{2\kappa} du.$$

It is shown in A* that $S_n(x)$ is a trigonometric sum in x of order not higher than $\kappa(m-1)$, and so not higher than n, and that $S_n(x)$ represents $\phi(x)$ with the required degree of approximation.

This information as to the form of $S_n(x)$ will not be enough for us here. We shall show that†

$$S_n(x) = \int_{-\pi}^{\pi} \phi(t)\, T_n(t-x)\, dt, \tag{39}$$

where

$$T_n(t-x) = \sum_{\nu=0}^{n} A_{n\nu} \cos \nu(t-x), \tag{40}$$

and the coefficients $A_{n\nu}$ are constants independent of ϕ. It will be sufficient to show that each of the k terms of which $S_n(x)$ is composed in (38) has this form. Consider any one of these terms, say that containing $\phi(x+2ru)$. The binomial coefficient and the factor $\pm h_m$ may be left out of account. The change of variable $t = x + 2ru$ gives

$$\int_{-\pi/2}^{\pi/2} \phi(x+2ru) \left[\frac{\sin mu}{m \sin u} \right]^{2\kappa} du = \frac{1}{2r} \int_{x-r\pi}^{x+r\pi} \phi(t) \left[\frac{\sin m \dfrac{t-x}{2r}}{m \sin \dfrac{t-x}{2r}} \right]^{2\kappa} dt.$$

Because of the periodicity of the functions involved, this is, except for the irrelevant constant factor $1/(2r)$, the same as

$$\int_{-r\pi}^{r\pi} \phi(t) \left[\frac{\sin m \dfrac{t-x}{2r}}{m \sin \dfrac{t-x}{2r}} \right]^{2\kappa} dt.$$

Let the interval of integration be broken up into r intervals, each of length 2π, and let these be reduced to a common interval by change of variable. In this way the last integral can be brought into the form

$$\int_{-r\pi}^{-(r-2)\pi} \phi(t) \sum_{j=0}^{r-1} \left[\frac{\sin m \left(\dfrac{t-x}{2r} + \dfrac{2j\pi}{2r} \right)}{m \sin \left(\dfrac{t-x}{2r} + \dfrac{2j\pi}{2r} \right)} \right]^{2\kappa} dt. \tag{41}$$

* The notation has been changed somewhat.

† Cf. B, pp. 347–348, where only the case $k = 1$ is taken up.

Now it is a familiar fact* that $[(\sin \frac{1}{2}mv)/(m \sin \frac{1}{2}v)]^2$ is a finite trigonometric sum in v, of order $m-1$, involving only cosines, and so, if this expression is raised to the κth power, there will be obtained a sum of the same character, of order $\kappa(m-1)$. That is, each term under the sign of summation in the integral just written down has the form

$$\sum_{i=0}^{\kappa(m-1)} B_i \cos i\left(\frac{t-x}{r} + \frac{2j\pi}{r}\right),$$

where the coefficients B_i are constants. On performing the summation with regard to j, a double sum is obtained which can be written thus:

$$(42)\qquad \sum_{i=0}^{\kappa(m-1)} \left\{ B_i \left[\cos i\frac{t-x}{r} \sum_{j=0}^{r-1} \cos\frac{2ij\pi}{r} - \sin i\frac{t-x}{r} \sum_{j=0}^{r-1} \sin\frac{2ij\pi}{r} \right] \right\}.$$

Since

$$\sum_{j=0}^{r-1} \cos\frac{2ij\pi}{r}$$

is zero unless i is divisible by r, and the corresponding sum of sines is always zero,† the expression (42) really involves no sines of multiples of $(t-x)/r$ at all, and the cosines of only such multiples of $(t-x)/r$ as are at the same time integral multiples of $t-x$. It has the same form as the right-hand side of (40), with the coefficients for which the second subscript is greater than n/r all equal to zero. As the integrand in (41), regarded as a function of t, is now seen to have the period 2π, the interval of integration may equally well be taken as that from $-\pi$ to π. To justify (39) and (40) it remains only to combine the terms corresponding to the several values of r.

The approximating property of $S_n(x)$, stated precisely, is this: If $\phi(x)$, of period 2π, has a $(k-1)$th derivative satisfying the condition (25), then, for all positive integral values of n and all values of x,

$$|\phi(x) - S_n(x)| \leqq \frac{K_k\,\mu}{n^k},$$

where K_k is a constant depending only on k.

Of course the number k enters into the definition of $S_n(x)$. If ϕ had a derivative of order $l-1$ satisfying a Lipschitz condition, $l > k$, it would not in general be true of this $S_n(x)$ that $|\phi(x) - S_n(x)|$ remains inferior to a constant multiple of $1/n^l$; to attain this degree of approximation, it would be necessary to define a new S_n by means of a new T_n. But it is readily seen, on following through the demonstration in A, that the function $T_n(t-x)$ formed for any particular value of k applies equally well for any *smaller*

* See Fejér, Mathematische Annalen, vol. 58 (1904), pp. 51–69; p. 53.

† See, e. g., Bôcher, *Introduction to the theory of Fourier's series*, Annals of Mathematics, ser. 2, vol. 7 (1906), pp. 81–152; p. 135.

value of k, and this is important for what follows. It will be well to give a summary of the facts that will be needed about the functions S_n and T_n in the form of a lemma.

LEMMA VI. *The positive integer k' being regarded as fixed, there exists for every positive integral value of n a cosine-sum $T_n(t-x)$, of the form* (40), *such that if $\phi(x)$ is any function of period 2π whose $(l-1)$th derivative, $l \leqq k'$, everywhere satisfies the condition*

$$|\phi^{(l-1)}(x_2) - \phi^{(l-1)}(x_1)| \leqq \mu |x_2 - x_1|,$$

then the corresponding $S_n(x)$, defined by (39), *satisfies for all values of x the relation*

$$|\phi(x) - S_n(x)| \leqq \frac{g\mu}{n^l},$$

where g depends only on k'.

For the purpose of obtaining a theorem on the approximate representation of a function which has a $(k-1)$th derivative satisfying a Lipschitz condition, by means of linear combinations of the functions $U_n(x)$, the number k' in the lemma is to be set equal to $k+1$, while l is to be given the values $k+1$ and k successively.

The former value is used in studying the behavior of the coefficients $A_{n\nu}$. Let $\cos \nu x$ be substituted for $\phi(x)$ in the lemma. It satisfies the relation

$$\left| \frac{d^k}{dx^k} \cos \nu x_2 - \frac{d^k}{dx^k} \cos \nu x_1 \right| \leqq \nu^{k+1} |x_2 - x_1|,$$

as is seen at once by applying the mean-value theorem to the kth derivative. Therefore

$$\left| \cos \nu x - \int_{-\pi}^{\pi} T_n(t-x) \cos \nu t \, dt \right| \leqq \frac{g\nu^{k+1}}{n^{k+1}}.$$

On the other hand, it follows from (40) that for $\nu \geqq 1$

$$\int_{-\pi}^{\pi} T_n(t-x) \cos \nu t \, dt = \pi A_{n\nu} \cos \nu x,$$

so that

$$|\cos \nu x - \pi A_{n\nu} \cos \nu x| \leqq \frac{g\nu^{k+1}}{n^{k+1}}$$

for all values of x, or, setting $x = 0$ in particular,

$$|1 - \pi A_{n\nu}| \leqq \frac{g\nu^{k+1}}{n^{k+1}}. \tag{43}$$

The same reasoning applied when $\nu = 0$ shows that

$$A_{n0} = \frac{1}{2\pi}$$

precisely.

It appears on substituting (40) in (39) that for any function $\phi(x)$

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(t)\,dt + \sum_{\nu=1}^{n}\left\{A_{n\nu}\left[\cos\nu x\int_{-\pi}^{\pi}\phi(t)\cos\nu t\,dt + \sin\nu x\int_{-\pi}^{\pi}\phi(t)\sin\nu t\,dt\right]\right\}$$

$$= a_0 + \sum_{\nu=1}^{n} d_{n\nu}(a_\nu\cos\nu x + b_\nu\sin\nu x),$$

where a_ν, b_ν are the Fourier coefficients of ϕ, and the numbers $d_{n\nu} = \pi A_{n\nu}$ are independent of ϕ. That is, the functions $S_n(x)$ are formed by applying a particular method of summation to the Fourier's series for ϕ. Rewritten in the new notation, (43) states that

$$|1 - d_{n\nu}| \leqq \frac{g\nu^{k+1}}{n^{k+1}}. \tag{44}$$

For the sake of uniformity we will introduce a coefficient $d_{n0} = 1$.

We are now in a position to define the approximating function desired. Let it be assumed that $\phi(x)$ and $\lambda(x)$ satisfy the hypotheses of Theorem II, and let α_n denote as before the general coefficient in the Sturm-Liouville series for $\phi(x)$. The function to be used is the following:

$$\Sigma_n(x) = \sum_{\nu=0}^{n} d_{n\nu}\,\alpha_\nu\,U_\nu(x).$$

It will be shown that as n becomes infinite $\Sigma_n(x)$ converges uniformly to a function $\psi(x)$, which is then necessarily continuous, so rapidly that

$$|\psi(x) - \Sigma_n(x)|$$

does not exceed a constant multiple of $1/n^k$. When this has been done, it will remain to be proved that $\psi(x)$ and $\phi(x)$ are identical.

Suppose $\phi(x)$ defined for values of x outside of the interval $(0, \pi)$ so as to be an even function of period 2π. This function will have everywhere a $(k-1)$th derivative satisfying a Lipschitz condition of the form (25), so that Lemma VI is applicable with $l = k$. Since ϕ is even, $S_n(x)$ now has the form

$$S_n(x) = \sum_{\nu=0}^{n} d_{n\nu}\,a_\nu\cos\nu x,$$

where

$$a_\nu = \frac{2}{\pi}\int_0^{\pi}\phi(t)\cos\nu t\,dt, \qquad \nu \geqq 1,$$

with a corresponding formula for a_0.

Consider the difference

$$\Sigma_{n+p}(x) - \Sigma_n(x),$$

where p is any positive integer. It is equal to

$$\begin{aligned}(45)\qquad & S_{n+p}(x) - S_n(x) + \sum_{\nu=1}^{n}(d_{n+p,\,\nu} - d_{n\nu})[\alpha_\nu U_\nu(x) - a_\nu \cos \nu x] \\ & \qquad + \sum_{\nu=n+1}^{n+p} d_{n+p,\,\nu}[\alpha_\nu U_\nu(x) - a_\nu \cos \nu x].\end{aligned}$$

By Lemma VI,

$$|S_{n+p}(x) - S_n(x)| = |[\phi(x) - S_n(x)] - [\phi(x) - S_{n+p}(x)]| \leqq \frac{2g\mu}{n^k}.$$

From (44),

$$|d_{n+p,\,\nu} - d_{n\nu}| \leqq \frac{2g\nu^{k+1}}{n^{k+1}},$$

and from (27), the correctness of which was established under the hypotheses that we are using now,

$$(46)\qquad |\alpha_\nu U_\nu(x) - a_\nu \cos \nu x| \leqq \frac{\beta}{\nu^{k+1}},$$

where β is a constant. Hence the absolute value of each term under the first sign of summation in (45) is less than or equal to $2g\beta/n^{k+1}$, and the absolute value of the sum of n terms does not exceed $2g\beta/n^k$. As for the other sum, it follows from (44) that

$$|d_{n+p,\,\nu}| \leqq 1 + g,$$

for all values of n, p, and ν, while (46) is still satisfied, and so

$$\begin{aligned}&\left|\sum_{\nu=n+1}^{n+p} d_{n+p,\,\nu}[\alpha_\nu U_\nu(x) - a_\nu \cos \nu x]\right| \\ &\qquad \leqq \beta(1+g)\sum_{\nu=n+1}^{n+p}\frac{1}{\nu^{k+1}} < \beta(1+g)\int_n^\infty \frac{dt}{t^{k+1}} = \frac{\beta(1+g)}{kn^k}.\end{aligned}$$

Combining these facts, we see that the absolute value of $\Sigma_{n+p}(x) - \Sigma_n(x)$ does not exceed $1/n^k$ multiplied by a quantity independent of x, n, and p. Consequently $\Sigma_n(x)$ does uniformly approach a continuous limiting function $\psi(x)$ as n becomes infinite, and

$$\psi(x) = \Sigma_n(x) + O\left(\frac{1}{n^k}\right)$$

uniformly for all values of x in the interval from 0 to π.

There would be no difficulty in showing that if $\phi^{(k-1)}(0) = 0$, the absolute value of $\psi(x) - \Sigma_n(x)$ does not exceed $c\mu/n^k$, where c is independent of ϕ; but as in other cases of the same sort we will not go into detail on this point.

It is still to be proved that the limiting function $\psi(x)$ is identical with $\phi(x)$. It might be interesting to know whether it is true that a convergent series will always be summable by means of the factors $d_{n\nu}$, and that the value so obtained will always be equal to the sum of the series. We shall leave this general question aside, and confine our attention to the special problem in hand, making use of the fact that $s_n(x)$, $S_n(x)$, and $\sigma_n(x)$ all converge to the value $\phi(x)$. The proof is as follows:

Let x be any number in the interval $0 \leqq x \leqq \pi$.

Let ϵ be any positive quantity.

Let $S_n^q(x)$ denote the sum of the first $q+1$ terms of $S_n(x)$, and $\Sigma_n^q(x)$ the sum of the first $q+1$ terms of $\Sigma_n(x)$, when $n \geqq q$.

Two integers n and q are to be chosen subject to seven conditions; the possibility of fulfilling these conditions will be made clear after the conditions themselves have been written down.

Let q be a number such that

(47) $$|\phi(x) - s_q(x)| < \epsilon/7,$$

(48) $$|\sigma_q(x) - \phi(x)| < \epsilon/7,$$

(49) $$|[\Sigma_n(x) - \Sigma_n^q(x)] - [S_n(x) - S_n^q(x)]| < \epsilon/7$$

for every value of $n > q$. Let q, once chosen, be held fast, and let a number $n > q$ be chosen so that

(50) $$|s_q(x) - S_n^q(x)| < \epsilon/7,$$

(51) $$|\Sigma_n^q(x) - \sigma_q(x)| < \epsilon/7,$$

(52) $$|S_n(x) - \phi(x)| < \epsilon/7,$$

(53) $$|\psi(x) - \Sigma_n(x)| < \epsilon/7.$$

That (47), (48), (52), and (53) can be satisfied, is an immediate consequence of the convergence of the respective sums involved. In (50),

$$s_q(x) - S_n^q(x) = \sum_{\nu=1}^{q} (1 - d_{n\nu}) a_\nu \cos \nu x,$$

and for a fixed value of q the right-hand member involves n only in the differences $1 - d_{n\nu}$, and approaches zero as n becomes infinite. Similar reasoning applies to (51). In the remaining condition (49),

$$\Sigma_n(x) - \Sigma_n^q(x) = \sum_{\nu=q+1}^{n} d_{n\nu}\, \alpha_\nu\, U_\nu(x),$$

$$S_n(x) - S_n^q(x) = \sum_{\nu=q+1}^{n} d_{n\nu}\, a_\nu \cos \nu x,$$

and the difference of these expressions is, except as to notation, the second sum in (45), and is inferior in absolute value to a quantity which is independent of n and approaches zero as q becomes infinite. Hence (49) also can be fulfilled. Of course it will be essential that all seven conditions be satisfied simultaneously, but as each by itself is satisfied for *all* values of q or of n from some point on, there will be no difficulty in finding a single value of q for the first three, and then a single value of n for the last four.

With the justification of the seven inequalities, the proof is practically complete. It is found upon adding that the sum of the quantities inclosed in bars on the left-hand sides is simply $\psi(x) - \phi(x)$, and as the absolute value of this sum can not exceed the sum of the absolute values, it follows that

$$|\phi(x) - \psi(x)| < \epsilon$$

for suitable values of q and n. But ϕ and ψ are independent of q and n, and as ϵ is arbitrarily small, it must be that $\psi(x) = \phi(x)$ exactly. As this has been proved for a value of x which is any value in $(0, \pi)$, the equation is an identity. The conclusion is as follows:

THEOREM IV. *If $\phi(x)$ and $\lambda(x)$ satisfy the hypotheses of Theorem II, there exists for every positive integral value of n a linear combination $\Sigma_n(x)$ of the functions $U_\nu(x)$, $\nu = 0, 1, \cdots, n$, with constant coefficients, such that*

$$\phi(x) = \Sigma_n(x) + O\left(\frac{1}{n^k}\right)$$

uniformly throughout the interval $0 \leqq x \leqq \pi$.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.